PARABOLIC GEOMETRIES FOR PEOPLE THAT LIKE PICTURES

LECTURE 4 WARM-UP: A PRIMER ON $(\mathrm{SL}_2\,\mathbb{R}, B)$

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Consider the Lie group $\mathrm{SL}_2\,\mathbb{R}$ of linear transformations of $\mathbb{R}^2$ with determinant 1, which we can represent via matrices as

$$\mathrm{SL}_2\,\mathbb{R} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\},$$

and the *Borel subgroup*

$$B := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$

By definition, we have a natural action of $\mathrm{SL}_2\,\mathbb{R}$ on $\mathbb{R}^2$, and this induces an action on the projective line $\mathbb{RP}^1$, the space of 1-dimensional linear subspaces of $\mathbb{R}^2$. Under this action, the subgroup B is the stabilizer of the point $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathbb{RP}^1$ corresponding to the line generated by $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, so since $\mathrm{SL}_2\,\mathbb{R}$ acts transitively on $\mathbb{RP}^1$, we may identify the homogeneous space $\mathrm{SL}_2\,\mathbb{R}/B$ with $\mathbb{RP}^1$.

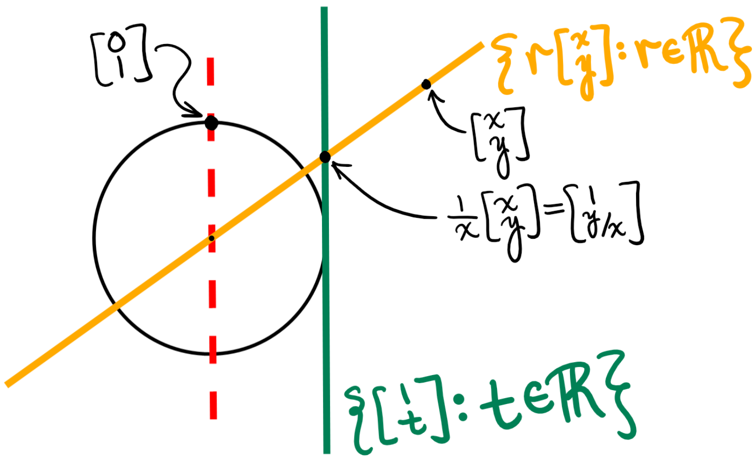

Figure 1. Every 1-dimensional subspace $\langle\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\rangle$ of $\mathbb{R}^2$ intersects the line $\{\left[\begin{smallmatrix} 1 \\ t \end{smallmatrix}\right] : t \in \mathbb{R}\}$, except for the line $\langle\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\rangle$

Each 1-dimensional subspace of $\mathbb{R}^2$ intersects the unit circle at a unique pair of antipodal points, so we can think of $\mathbb{RP}^1$ as the circle with antipodal points identified. Moreover, all but one point of $\mathbb{RP}^1$ is of the form $\left(\begin{smallmatrix}1\\t\end{smallmatrix}\right)$ for some $t \in \mathbb{R}$, so we can also think of $\mathbb{RP}^1$ as a copy of the affine line together with a "point at infinity" $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

As transformations, the one-parameter subgroup $\exp\left(s\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}1&0\\s&1\end{smallmatrix}\right]$ acts by translations on the affine line and fixes the point at infinity:

$$\begin{bmatrix}1&0\\s&1\end{bmatrix}\cdot\begin{pmatrix}1\\t\end{pmatrix}=\begin{pmatrix}1\\s+t\end{pmatrix} \text{ and } \begin{bmatrix}1&0\\s&1\end{bmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}.$$

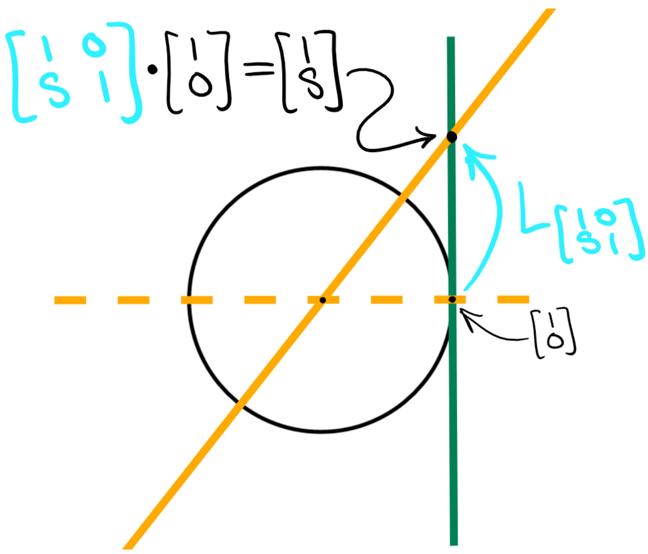

FIGURE 2. The transformation $\left[\begin{smallmatrix}1&0\\s&1\end{smallmatrix}\right]$ acts by translating along the copy of the affine line

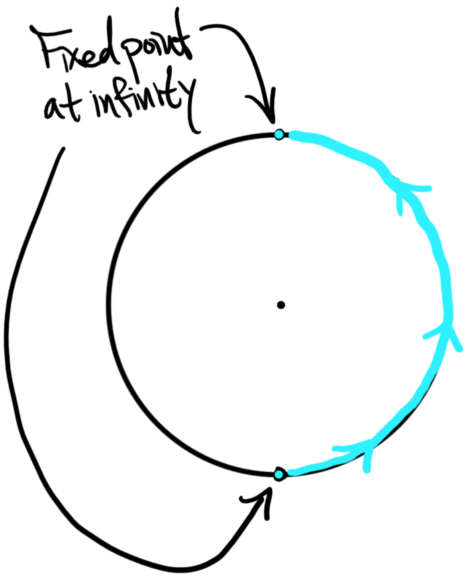

FIGURE 3. A depiction of the left-action of $\left[\begin{smallmatrix}1&0\\s&1\end{smallmatrix}\right]$ on the circle with antipodal points identified

The one-parameter subgroup $\exp\left(s\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}e^s & 0\\ 0 & e^{-s}\end{smallmatrix}\right]$, on the other hand, acts by rescaling the affine line by e^{-2s}, and also fixes the point at infinity:

$$\begin{bmatrix}e^s & 0\\ 0 & e^{-s}\end{bmatrix} \cdot \begin{pmatrix}1\\ t\end{pmatrix} = \begin{pmatrix}e^s\\ e^{-s}t\end{pmatrix} = \begin{pmatrix}1\\ e^{-2s}t\end{pmatrix}$$

and

$$\begin{bmatrix}e^s & 0\\ 0 & e^{-s}\end{bmatrix} \cdot \begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}0\\ e^{-s}\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

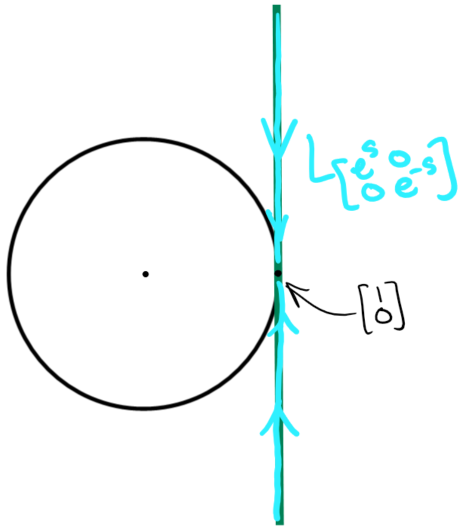

FIGURE 4. The transformation $\left[\begin{smallmatrix}e^s & 0\\ 0 & e^{-s}\end{smallmatrix}\right]$ acts by rescaling the copy of the affine line by e^{-2s}

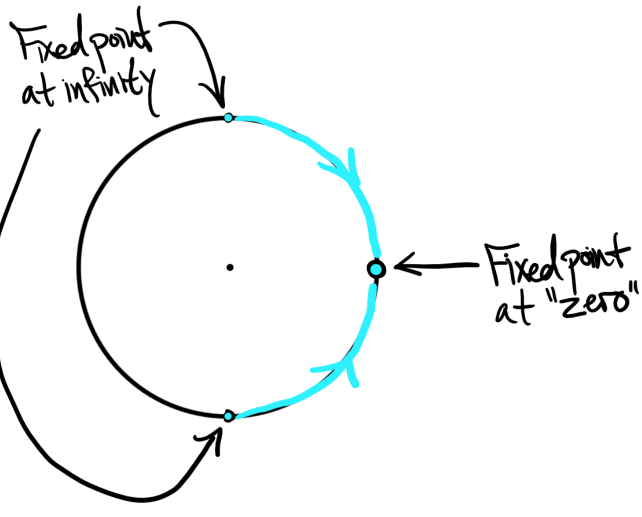

FIGURE 5. A depiction of the left-action of $\left[\begin{smallmatrix}e^s & 0\\ 0 & e^{-s}\end{smallmatrix}\right]$ on the circle with antipodal points identified

Finally, we have the one-parameter subgroup $\exp\left(s\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}1&s\\0&1\end{smallmatrix}\right]$, which is a bit of an oddity. The transformation $\left[\begin{smallmatrix}1&s\\0&1\end{smallmatrix}\right]$ fixes the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, as the rescaling matrix did, but not the "point at infinity". Instead, it acts by translations along another copy of the affine line, given by $\{\left(\begin{smallmatrix}t\\1\end{smallmatrix}\right) : t \in \mathbb{R}\}$; with respect to this new affine line, the point $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ would be thought of as the "point at infinity":

$$\begin{bmatrix}1 & s\\0 & 1\end{bmatrix} \cdot \begin{pmatrix}t\\1\end{pmatrix} = \begin{pmatrix}s+t\\1\end{pmatrix} \text{ and } \begin{bmatrix}1 & s\\0 & 1\end{bmatrix} \cdot \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

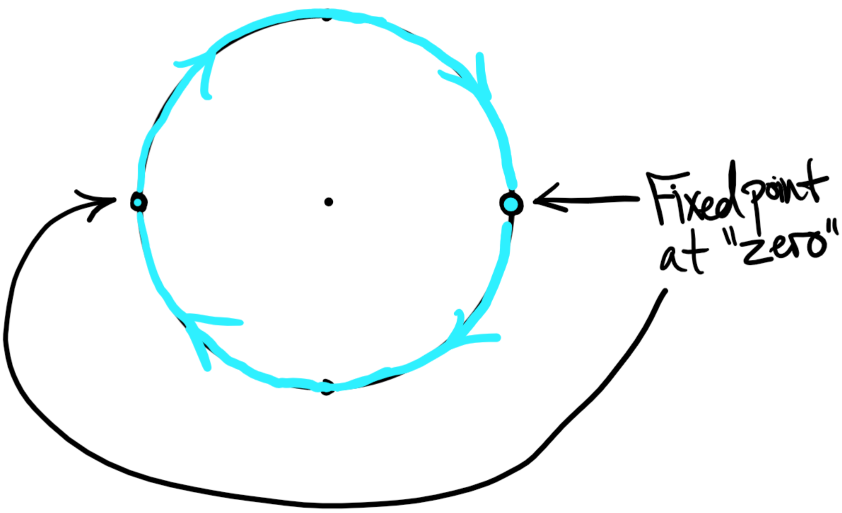

FIGURE 6. A depiction of the left-action of $\left[\begin{smallmatrix}1&s\\0&1\end{smallmatrix}\right]$ on the circle with antipodal points identified

On the original copy of the affine line, we have

$$\begin{bmatrix}1 & s\\0 & 1\end{bmatrix} \cdot \begin{pmatrix}1\\t\end{pmatrix} = \begin{pmatrix}1+st\\t\end{pmatrix},$$

which corresponds to the point $\frac{t}{1+st} \in \mathbb{R}$ when $1 + st \neq 0$. I've gotten in the habit of calling these things "(unipotent) tilts", for reasons that will be clearer in a moment, though there isn't really a standard terminology for these transformations.

An additional, and convenient, one-parameter subgroup for this geometry is $\exp\left(\theta\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{smallmatrix}\right]$, which corresponds to just rotating the circle (while keeping antipodal points identified). The reason that this one-parameter subgroup is convenient here is that it happens to act transitively on $\mathrm{SL}_2\,\mathbb{R}/B$, since it just rotates along the circle.

As I explained in the last lecture, we think of $\mathrm{SL}_2\,\mathbb{R}$ as the space of configurations of an observer over $\mathrm{SL}_2\,\mathbb{R}/B \cong \mathbb{RP}^1$. Each matrix $A \in \mathrm{SL}_2\,\mathbb{R}$ uniquely determines a pair of column vectors $A\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right)$ and $A\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)$. The vector $A\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right)$ generates the line $\left\langle A\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right)\right\rangle$ corresponding to $q_B(A)$, and $A\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)$ determines the copy of the affine line along which we move when we right-translate by $\left[\begin{smallmatrix}1&0\\s&1\end{smallmatrix}\right]$. Indeed, for $[u\ v] \in \mathrm{SL}_2\,\mathbb{R}$, we have

$$[u\ v]\begin{bmatrix}1 & 0\\ s & 1\end{bmatrix} = [u + sv\ v],$$

so when we right-translate $[u\ v]$ by $\left[\begin{smallmatrix}1&0\\s&1\end{smallmatrix}\right]$, we move from the point $q_B([u\ v])$ corresponding to the line $\langle u\rangle$ to the point $q_B([u+sv\ v])$ corresponding to the line $\langle u + sv\rangle$.

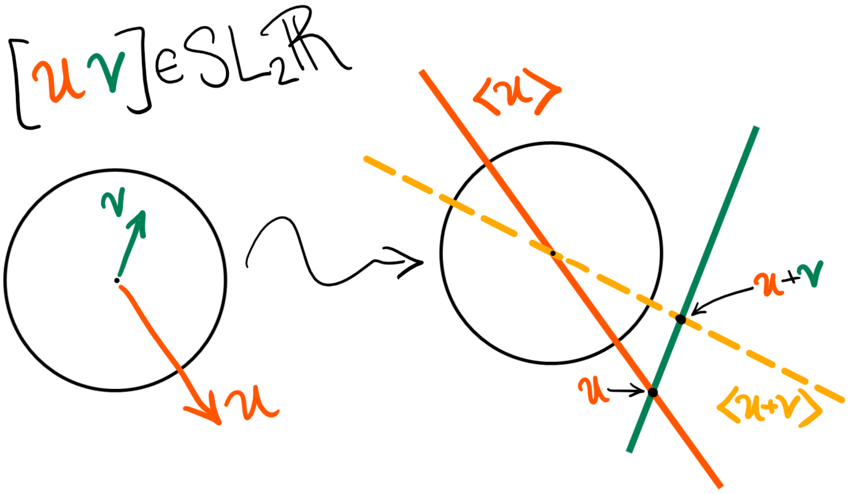

FIGURE 7. In the matrix $[u\ v] \in \mathrm{SL}_2\,\mathbb{R}$, the vector u determines the line $\langle u\rangle$ corresponding to the point $q_B([u\ v])$ and v determines a copy of the affine line along which we move when we right-translate by $\left[\begin{smallmatrix}1&0\\s&1\end{smallmatrix}\right]$

The rescalings, tilts, and $-\mathbb{1}$ together generate the subgroup B, so given a configuration $g \in \mathrm{SL}_2\,\mathbb{R}$ over a point $q_B(g) \in \mathrm{SL}_2\,\mathbb{R}/B$, changing to a different configuration over that point corresponds to right-translating by compositions of these tilts, rescalings, and $-\mathbb{1}$. As with affine geometry, it is worth taking a moment to imagine what this looks like.

As we might expect, right-translating A by a rescaling just rescales ourselves along the affine line determined by $A\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)$. The unipotent tilts, on the other hand, *tilt* the affine line determined by $A\left(\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)$ along the line corresponding to $q_B(A)$:

$$[u\ v]\begin{bmatrix}1 & s\\ 0 & 1\end{bmatrix} = [u\ su + v].$$

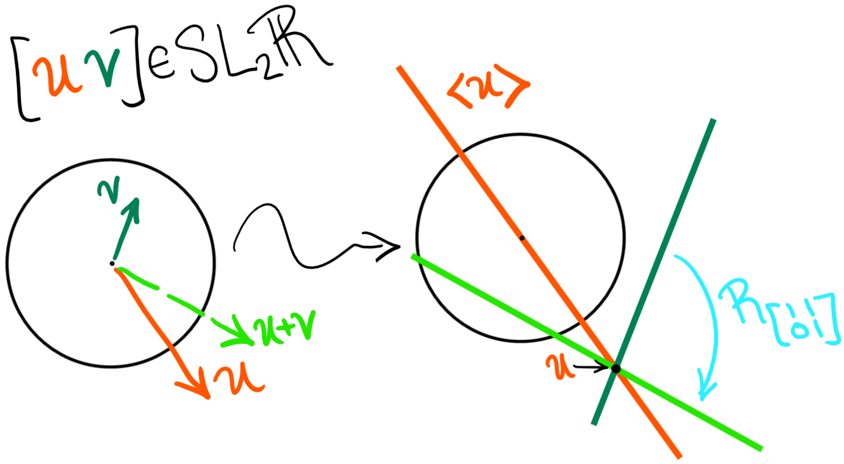

FIGURE 8. Right-translating $[u\ v] \in \mathrm{SL}_2\,\mathbb{R}$ by a unipotent tilt takes the affine line determined by v and tilts it along the line determined by u